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On the Degree of Approximation with Hermite Interpolatory Side Conditions

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1. INTRODUCTION

The aim of this paper is to compare the degree of uniform approximation of a function $f \in C^{\kappa}[-1, 1]$ by algebraic polynomials of degree ν, $E_\nu(f)$, to the degree of uniform approximation when the polynomials are restricted to satisfy

$$p_\nu^{(j)}(t_i) = f^{(j)}(t_i), \qquad i = 1,..., \gamma, \qquad j = 0, 1,..., \kappa,$$

denoted $E_\nu(f, A_\kappa)$. Clearly

$$E_\nu(f, A_\kappa) \geqslant E_\nu(f), \qquad \forall \nu \geqslant \gamma(\kappa + 1) - 1.$$

Our goal is to obtain an "inverse" result. A special case of our result is

$$E_\nu(f) = O(\nu^{-\beta}) \Rightarrow E_\nu(f, A_\kappa) = O(\nu^{-\beta}) \tag{1}$$

unless β is an even integer, $\kappa < \beta \leqslant 2\kappa$, f is not in $C^{2\kappa}[-1, 1]$ and one of the interpolation nodes is ± 1, in which case

$$E_\nu(f) = O(\nu^{-\beta}) \Rightarrow E_\nu(f, A_\kappa) = O(\nu^{-\beta} \log \nu). \tag{2}$$

Our method of proof is to transform to the trigonometric case, find an even interpolant that approximates and interpolates $f(\cos \theta)$, and then transform back. As is usual, the difficulty comes at the endpoints, and, in this case, in order to transform the interpolation at the endpoints, it is necessary to interpolate to order 2κ in the trigonometric setting. The latter is what presents the difficulty and eventually leads to the difference in estimates (1) and (2).

Hill *et al.* [3] have proved a Jackson-type estimate for $E_\nu(f, A_\kappa)$ (see also Beatson [2, Theorem 2.4] for a different proof and for the following statement of the result).

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THEOREM 1. *For each $k = 1, 2, 3,\ldots,$ there exists an L_k, and for each set of side conditions A_κ with $\kappa \leq k$ there exists a ν_1, not depending on the function $f \in C^\kappa[-1, 1]$, such that $E_\nu(f, A_\kappa)$ is well defined and satisfies*

$$E_\nu(f, A_\kappa) \leq L_k \nu^{-k} \omega(f^{(k)}, \nu^{-1}), \forall \nu \geq \nu_1,$$

where $\omega(f^{(k)}, \cdot)$ is the modulus of continuity of $f^{(k)}$ on $[-1, 1]$.

In many cases the estimate of this paper will be stronger than that of Theorem 1. For example, Timan [6, pp. 342–343] shows that it $f'(x) = (1 - x^2)^{1/4}$, then $E_\nu(f') = O(\nu^{-1/2})$, implying $E_\nu(f) = O(\nu^{-3/2})$. At the same time $\nu^{-1}\omega(f', \nu^{-1})$ is of the exact order $\nu^{-5/4}$. For this function and approximation from polynomials in the set

$$A_1 = \{h \in C^1[-1, 1]; h(1) = f(1) \text{ and } h'_-(1) = f'_-(1)\}$$

Theorem 1 provides the estimate $E_\nu(f, A_\kappa) = O(\nu^{-5/4})$, whereas the result of this paper provides the estimate $E_\nu(f, A_\kappa) = O(\nu^{-3/2})$.

2. ESTIMATES IN TERMS OF $e_\nu(g^{(\kappa)})$ WHERE $g(\theta) = f(\cos \theta)$

Beatson [1] considered the problem of approximating a κ-times continuously differentiable 2π periodic function g (henceforth written $g \in C^{*\kappa}[-\pi, \pi]$) by trigonometric polynomials satisfying Hermite interpolatory side conditions. Define A_κ^*, $E_\kappa^*(g)$, $E_\kappa^*(g, A_\kappa^*)$, similarly to A_κ, $E_\nu(f)$, $E_\nu(f, A_\kappa)$, but with nodes of interpolation now in T, the unit circle, and uniform approximation by trigonometric polynomials on T. Also $e_\nu(g^{(\kappa)})$ denotes the degree of approximation of $g^{(\kappa)}$ by trigonometric polynomials of degree ν (at most) with constant part zero. Then

THEOREM 2 [1, Theorem 2.1]. *For each $\kappa = 1, 2, 3,\ldots$. There exists an $M_\kappa > 0$, and for each set of side conditions A_κ^* a $\nu_1 = \nu_1(\kappa, t_1,\ldots, t_\gamma)$ not depending on g such that for any $g \in C^{*\kappa}[-\pi, \pi]$ $E_\kappa^*(g, A_\kappa^*)$ is defined and satisfies*

$$E_\nu^*(g, A_\kappa^*) \leq M_\kappa \nu^{-\kappa} e_\nu(g^{(\kappa)}), \qquad \forall \nu \geq \nu_1.$$

As a corollary to this theorem we have

COROLLARY 3 [1, Corollary 2.4]. *For each $\kappa = 1, 2, 3,\ldots,$ there exists an $M_\kappa > 0$, and for each set of side conditions A_κ provided that $-1 < t_i \leq 1$.*

$i = 1,\ldots, \gamma$, a ν_1, not depending on f, such that for any $f \in C^\kappa[-1, 1]$ $E_\nu(f, A_\kappa)$ is defined and satisfies

$$E_\nu(f, A_\kappa) \leqslant M_\kappa \nu^{-\kappa} e_\nu(g^{(\kappa)}), \qquad \forall \nu \geqslant \nu_1,$$

where $g \in C^{*\kappa}[-\pi, \pi]$ is defined by $g(\theta) = f(\cos\theta)$.

This corollary follows upon use of the standard transformation $g(\theta) = f(\cos\theta)$. Crucial to its proof is that for $f_1, f_2 \in C^\kappa[-\pi, \pi]$ and $|t_i| < 1$, the condition

$$\frac{d^j f_1}{dx^j}(t_i) = \frac{d^j f_2}{dx^j}(t_i), \qquad j = 0,\ldots, \kappa,$$

is equivalent to

$$\frac{d^j g_1}{d\theta^j}(\theta_{2i}) = \frac{d^j g_2}{d\theta^j}(\theta_{2i}), \qquad \frac{d^j g_1}{d\theta^j}(\theta_{2i-1}) = \frac{d^j g_2}{d\theta^j}(\theta_{2i-1}), \qquad j = 0,\ldots, \kappa;$$

where $\theta_{2i-1} = |\cos^{-1} t_i|$, $\theta_{2i} = -\theta_{2i-1}$, and $g_1(\theta) = f_1(\cos\theta)$, $g_2(\theta) = f_2(\cos\theta)$.

If $t_i = \pm 1$ is one of the interpolation nodes then the transformation of the interpolation conditions associated with the transformation $f \to g$ is less simple.

LEMMA 4. *Let $f \in C[-1, 1]$ be $2m$ times differentiable with respect to x at $x = 1$ $[x = -1]$. In order that the condition*

$$\frac{d^j f}{dx^j}(1) = r_j; \quad j = 0, 1,\ldots, m; \qquad \left[\frac{d^j f}{dx^j}(-1) = s_j; j = 0,\ldots, m;\right]$$

be satisfied it is necessary and sufficient that

$$\frac{d^{2j} g}{d\theta^{2j}}(0) = \sum_{i=0}^{j} a_{2j,i} r_i, \quad j = 0,\ldots, m; \quad \left[\frac{d^{2j} g}{d\theta^{2j}}(\pi) = \sum_{i=0}^{j} b_{2j,i} s_i, \; j = 0,\ldots, m;\right],$$

where $g(\theta) = f(\cos\theta)$, the $a_{2j,i}$ $[b_{2j,i}]$ are constants which do not depend on f, $a_{0,0} = 1$ $[b_{0,0} = 1]$, and $a_{2j,j} = (-1)^j (2j)!/(j!2^j)$, $j > 0$. $[b_{2j,j} = (2j)!/(j!2^j)$, $j > 0.]$

Remark. By the sentence "f is $2m$ times differentiable at $x = 1$" we mean that the definition of f can be extended to an open interval (a, b) containing 1 so that all the derivatives up to order $2m - 1$ are defined on (a, b) and $f^{(2m)}(1)$ exists. This is equivalent to another formulation involving ordinary derivatives on $(a, 1)$ and one-sided derivatives at 1.

Proof. Since f is $2m$ times differentiable with respect to x at $x = 1$, g is $2m$ times differentiable with respect to θ at $\theta = 0$. All odd-order derivatives of g are zero at $\theta = 0$ since g is even. Writing

$$\frac{dg}{d\theta} = -\frac{df}{dx}\cdot\sin\theta, \qquad \frac{d^2g}{d\theta^2} = \frac{d^2f}{dx^2}\cdot\sin^2\theta - \frac{df}{dx}\cdot\cos\theta, \text{ etc.,}$$

one finds

$$\frac{d^kg}{d\theta^k}(0) = \sum_{i=0}^{k} a_{k,i}\frac{d^if}{dx^i}(1), \qquad k = 0, 2, \dots, 2m;$$

where the $a_{k,i}$ do not depend on f and $a_{0,0} = 1$. For $i > 0$, consider the particular function $f(x) = (x-1)^i/i!$ with all derivatives except the ith zero at $x = 1$, and $(d^if/dx^i)(1) = 1$. The corresponding periodic function g is given by the everywhere-convergent power series

$$g(\theta) = (\cos\theta - 1)^i/i! = \left(-\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)^i\Big/ i!$$

This series is differentiable term by term with

$$\frac{d^ng}{d\theta^n}(0) = 0, \ n = 0,\dots, 2i-1; \qquad \text{and} \qquad \frac{d^{2i}g}{d\theta^{2i}}(0) = \frac{(-1)^i(2i)!}{i!\,2^i}.$$

Since i was any positive integer, it follows that

$$\begin{aligned} & a_{n,i} = 0, && \forall(n,i)\in\{(n,i): 0 < i \leq n < 2i\} \\ \Rightarrow\ & a_{n,i} = 0, && \forall(n,i)\in\{(n,i): 0 < n \text{ and } n < 2i \leq 2n\} \\ \Rightarrow\ & a_{2k,j} = 0, && 0 < k < j \leq 2k; \end{aligned}$$

while from above $a_{2k,k} = (-1)^k\,(2k)!/(k!\,2^k)$, $k > 0$.

This shows the necessity of the condition for $x = 1$. Since the $(m+1)\times(m+1)$ lower-triangular matrix, with jth row

$$a_{2j,0}a_{2j,1}\cdots a_{2j,j}0\cdots 0,$$

is invertible, the condition is also sufficient. The case $x = -1$ may be treated similarly.

Lemma 4 and the discussion above it show that if $f \in C^{2\kappa}[-1, 1]$ and the even trigonometric polynomial $\tilde{t}_\nu$ interpolates to $g(\theta) = f(\cos\theta)$ and its first 2κ derivatives at $\theta_i = \cos^{-1}t_i$, $i = 1,\dots,\gamma$; then $p_\nu(x) = \tilde{t}_\nu(\cos^{-1}x)$ is an algebraic polynomial interpolating to f and its first κ derivatives at t_i, $i = 1,\dots,\gamma$. Hence arguing as in [1, Corollary 2.4] Theorem 2 implies

COROLLARY 5. *For each $\kappa = 1, 2, 3,\ldots$, there exists an $M_{2\kappa} > 0$; and for each set of side conditions A_κ, a ν_1, not depending on $f \in C^{2\kappa}[-1, 1]$; such that $E_\nu(f, A_\kappa)$ exists and satisfies*

$$E_\nu(f, A_\kappa) \leqslant M_{2\kappa}\nu^{-2\kappa}e_\nu(g^{(2\kappa)}), \qquad \forall \nu \geqslant \nu_1,$$

where $g \in C^{*2\kappa}[-\pi, \pi]$ is defined by $g(\theta) = f(\cos\theta)$.

3. THE MAIN RESULT

In what follows let N denote the set of natural numbers. Combining some results of Steckin [5] and Zamansky [7] we have

THEOREM 6. *For each $j \in N$ there exist constants A_j and B_j with the following properties: Let $g \in C^*[-\pi, \pi]$ and let t_ν be a sequence of polynomials (t_ν is of degree not exceeding ν) approximating g with $\|g - t_\nu\|$ non-increasing. Then*

(i) *$\sum_{n=1}^{\infty} n^{j-1}\|g - t_n\| < +\infty$ implies $g^{(j)}$ exists and is continuous, and*

$$\|g^{(j)} - t_\nu^{(j)}\| \leqslant A_j \sum_{n=[\nu/2]}^{\infty} n^{j-1}\|g - t_n\|, \qquad \forall \nu > 0.$$

(ii) *$\|t_\nu^{(j)}\| \leqslant B_j \sum_{n=1}^{\nu} n^{j-1}\|g - t_{n-1}\|$, $\forall \nu$.*

Proof. For a proof of the first statement see Lorentz [4, pp. 58–62].

To verify the second statement one starts with the Zamansky-type representation $(2^k \leqslant \nu < 2^{k+1}, j \in N)$

$$\|t_\nu^{(j)}\| \leqslant \|t_\nu^{(j)} - t_{2^k}^{(j)}\| + \sum_{i=1}^{k} \|t_{2^i}^{(j)} - t_{2^{i-1}}^{(j)}\| + \|t_1^{(j)} - t_0^{(j)}\|$$

and uses Bernstein's inequality, finding

$$\begin{aligned}
(1/2)\|t_\nu^{(j)}\| &- (\nu^j\|g - t_{2^k}\| + 2^j\|g - t_1\| + \|g - t_0\|) \\
&\leqslant \sum_{i=2}^{k} 2^{ji}\|g - t_{2^{i-1}}\| \leqslant 4\sum_{i=2}^{k}\left[2^{i(j-1)} \sum_{n=2^{i-2}+1}^{2^{i-1}} \|g - t_{2^{i-1}}\|\right] \\
&\leqslant 2^{2j} \sum_{n=2}^{2^{k-1}} n^{j-1}\|g - t_n\|,
\end{aligned}$$

implying

$$\|t_\nu^{(j)}\| \leqslant B_j \sum_{n=1}^{\nu} n^{j-1}\|g - t_{n-1}\|,$$

where B_j is a constant depending on j alone.

LEMMA 7. *Let μ be a nonnegative integer. There exists a constant D_μ, depending on μ only, and corresponding to any ψ distinct points $t_1, \ldots, t_\psi$ of $[-\pi, \pi)$ an integer $\nu_1 = \nu_1(\mu, t_1, \ldots, t_\psi)$ such that: For $\nu \geq \nu_1$ and arbitrary real numbers c_{ij} there is a trigonometric polynomial p_ν of degree $\leq \nu$ satisfying*

$$p_\nu^{(j)}(t_i) = \nu^j c_{ij}, \qquad i = 1, \ldots, \psi, \qquad j = 0, \ldots, \mu,$$

and

$$\| p_\nu \| \leq D_\mu \max_{i,j} | c_{ij} |.$$

Proof. Let T be the unit circle, and $B_1, \ldots, B_\psi$ be disjoint open sets in T containing $t_1, \ldots, t_\psi$. As in [1, Theorem 2.1] we can then construct for all ν not less than some ν_2 trigonometric polynomials h_{ij}, $i = 1, \ldots, \psi$, $j = 0, \ldots, \mu$ of degree at most ν with

$$\| h_{ij}^{(k)} \| \leq \nu^k, \qquad k = 0, 1, 2, \ldots,$$

$$h_{ij}^{(r)}(t_e) = 0, \qquad r = 0, \ldots, \mu, \quad i \neq e,$$

$$h_{ij}^{(r)}(t_i) = 0, \qquad r < j \quad \text{and} \quad h_{ij}^{(j)}(t_i) = j!\, \lambda^j,$$

where $\lambda = [\nu/(\mu + 1)]$ (≥ 1 for $\nu \geq \nu_2$), $[\cdot]$ is the integral part function, and

$$\| h_{ij} \|_{T \setminus B_i} \leq \delta_\nu, \qquad \text{where} \qquad \delta_\nu \to 0 \text{ as } \nu \to \infty.$$

We proceed to define some polynomials H_{ij} of degree $\leq \nu$ with the property that

$$H_{ij}^{(r)}(t_e) = \delta_{ie} \delta_{jr} \nu^j \qquad i, e = 1, \ldots, \psi; \qquad j, r = 0, \ldots, \mu.$$

Let i be arbitrary but fixed and define $H_{ij} = \sum_{e=0}^{\mu} b_e h_{ie}$, where the b_e are the solution of the equation

$$\begin{bmatrix} h_{i0}(t_i) & 0 & & & 0 \\ h_{i0}^{(1)}(t_i) & h_{i1}^{(1)}(t_i) & 0 & & 0 \\ h_{i0}^{(j)}(t_i) & h_{i1}^{(j)}(t_i) & \cdots\cdots & h_{ij}^{(j)}(t_i) & \\ h_{i0}^{(\mu)}(t_i) & h_{i1}^{(\mu)}(t_i) & \cdots\cdots\cdots\cdots & & h_{i\mu}^{(\mu)}(t_i) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_j \\ b_\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \nu^j \\ 0 \end{bmatrix}$$

For all $r = 0, \ldots, \mu$, divide the rth row of the matrix and the rth element of the product vector by $r!\lambda^r$. The matrix equation becomes $Ab = c$, where A is lower triangular and has ones on its diagonal. Since A has determinant one a unique solution exists. Also, since $\lambda \geq 1$, all the elements in A and c are bounded in magnitude by $(2(\mu + 1))^\mu$. Hence by Cramer's rule there exists an E_μ, depending on μ alone, such that $| b_j | \leqslant E_\mu$, $j = 0,..., \mu$. Hence

$$\| H_{ij} \|_{B_i} \leqslant F_\mu \qquad \text{and} \qquad \| H_{ij} \|_{T\setminus B_i} \leqslant \delta_\nu F_\mu ,$$

$j = 0,..., \mu$, where $F_\mu = E_\mu(\mu + 1)$. Since i was arbitrary this is also true for $i = 1,..., \psi$. Now choose $\nu_1 \geqslant \nu_2$ so large that $\delta_\nu \leqslant 1/\psi$ for $\nu \geqslant \nu_2$. Then

$$\| H_{ij} \|_{B_i} \leqslant F_\mu , \qquad \| H_{ij} \|_{T\setminus B_i} \leqslant F_\mu/\psi$$

for $i = 1,..., \psi, j = 0,..., \mu, \nu \geqslant \nu_1$. The polynomial

$$p_\nu = \sum_{i=1}^{\psi} \sum_{j=0}^{\mu} c_{ij} H_{ij}$$

has the properties listed in the statement of the lemma, with $D_\mu = 2(\mu - 1) F_\mu$.

THEOREM 8. *Let κ be a positive integer. There exist constants C_1 and C_2 depending on κ alone with the following property: Let $t_1, t_2,..., t_\gamma$, be distinct points in $[-1, 1]$ and $f \in C^\kappa[-1, 1]$. Suppose $\sum_{n=1}^{\infty} n^{\kappa-1}E_n(f) < \infty$ and let k be the largest integer in $[\kappa, 2\kappa]$ for which $\sum_{n=1}^{\infty} n^{k-1}E_n(f) < \infty$. Then there exists an integer $\nu_1 = \nu_1(\kappa, t_1,..., t_\gamma)$ such that for $\nu \geqslant \nu_1$*

$$E_\nu(f, A_\kappa) \leqslant C_1 \nu^{-k} \sum_{n \geqslant [\nu/2]} n^{k-1} E_n(f) + C_2 \sum_{k < 2j \leqslant 2\kappa} \nu^{-2j} \left(\sum_{n=1}^{\nu} n^{2j-1} E_{n-1}(f) \right)$$

where the term involving C_2 may be deleted unless ± 1 is one of the interpolation nodes.

Remark. The theorem gives no information about functions in $C^\kappa[-1, 1]$ for which $\sum_{n=1}^{\infty} n^{\kappa-1}E_n(f)$ diverges. However the "gap" is small. For example, $f \in C^\kappa[-1, 1]$ implies $E_n(f) = o(n^{-\kappa})$ and $E_n(f) = O(n^{-\kappa}(\log n)^{-2})$ implies that the series converges.

Proof of Theorem 8. Assume throughout the following that $\nu \geqslant \gamma(\kappa + 1) - 1$. If P is a polynomial of degree $\leqslant \nu$ then $E_\nu(f) = E_\nu(f - P)$, and $E_\nu(f, A_\kappa) = E_\nu(f - P, A_\kappa)$. In particular we may choose P so that $(f - P)^{(j)}(t_i) = 0$, $i = 1,..., \gamma$, $j = 0,..., \kappa$. Hence it suffices to prove the theorem when

$$f^{(j)}(t_i) = 0, \qquad i = 1,..., \gamma, \qquad j = 0,..., \kappa. \tag{3}$$

Case 1. One of the nodes of interpolation is ± 1. Assume without loss of generality that both -1 and $+1$ are nodes of interpolation. Let $g(\theta) = f(\cos\theta)$. Then $E_\nu(f) = E_\nu^*(g)$. Let

$$-\pi = -\theta_\gamma < \cdots < -\theta_2 < \theta_1 = 0 < \theta_2 < \cdots < \theta_\gamma = \pi$$

be the images of $t_1, \ldots, t_\gamma$ under the transformation $\theta = \pm\operatorname{arc\,cos}(t)$. Note that (3), the eveness of g, Lemma 4, and the discussion above it imply

$$g^{(j)}(\pm\theta_i) = 0, \qquad j = 0, \ldots, \kappa, \qquad i = 1, \ldots, \gamma. \tag{4}$$

For each ν let t_ν be the trigonometric polynomial of best uniform approximation to g. By Lemma 7 there exists a $D_{2\kappa}$ depending on κ alone, and a $\nu_1(\geqslant\gamma(2\kappa-1)-1)$ depending on κ and the nodes of interpolation, such that for $\nu \geqslant \nu_1$ there is a trigonometric polynomial of degree $\leqslant\nu$ satisfying

$$\begin{aligned} p_\nu^{(j)}(\pm\theta_i) &= (g - t_\nu)^{(j)}(\pm\theta_i), & 0 \leqslant j \leqslant k, \quad i = 1, \ldots, \gamma, \\ p_\nu^{(2j)}(\pm\theta_i) &= -t_\nu^{(2j)}(\pm\theta_i), & k < 2j \leqslant 2\kappa, \quad i \in \{1, \gamma\}, \end{aligned} \tag{5}$$

and having all other derivatives up to order 2κ zero at the nodes, with

$$\|p_\nu\| \leqslant D_{2\kappa}\Big(E_\nu(f) + \sum_{j=1}^{k} \nu^{-j} \|g^{(j)} - t_\nu^{(j)}\| + \sum_{k<2j\leqslant 2\kappa} \nu^{-2j} \|t_\nu^{(2j)}\|\Big).$$

Using the estimates of the quantities on the right-hand side above from Theorem 6 we find there exist constants A_1 and C_2 depending on κ only such that

$$\|p_\nu\| \leqslant A_1\nu^{-k} \sum_{n\geqslant[\nu/2]} n^{k-1}E_n(f) + C_2 \sum_{k<2j\leqslant 2\kappa} \nu^{-2j} \sum_{1}^{\nu} n^{2j-1}E_{n-1}(f). \tag{6}$$

Now consider the trigonometric polynomial $(t_\nu - p_\nu)$. Since $k \geqslant \kappa$, (4) and (5) imply

$$\begin{aligned} (t_\nu - p_\nu)^{(j)}(\pm\theta_i) &= 0, & j = 0, \ldots, \kappa, \quad & i = 1, \ldots, \gamma, \\ (t_\nu - p_\nu)^{(2j)}(\pm\theta_i) &= 0, & \kappa < 2j \leqslant 2\kappa, \quad & i \in \{1, \gamma\}. \end{aligned}$$

Since the nodes of interpolation are symmetric about zero the even part $(\tilde{t}_\nu - \tilde{p}_\nu)$ of this trigonometric polynomial also satisfies these conditions. In addition all the odd order derivatives of $(\tilde{t}_\nu - \tilde{p}_\nu)$ vanish at 0 and $\pm\pi$. Thus

$$\begin{aligned} (\tilde{t}_\nu - \tilde{p}_\nu)^{(j)}(\pm\theta_i) &= 0, & j = 0, \ldots, \kappa, \quad & i = 1, \ldots, \gamma, \\ (\tilde{t}_\nu - \tilde{p}_\nu)^{(j)}(\pm\theta_i) &= 0, & \kappa < j \leqslant 2\kappa, \quad & i \in \{1, \gamma\}. \end{aligned} \tag{7}$$

Since g is even and using (6)

$$\| g - (\tilde{t}_\nu + \tilde{p}_\nu) \| \leqslant \| g - (t_\nu + p_\nu) \| \leqslant \| g - t_\nu \| + \| p_\nu \|$$

$$\leqslant C_1 \nu^{-k} \sum_{n=[\nu/2]}^{\infty} n^{k-1} E_n(f) + C_2 \sum_{k \leqslant 2j < 2\kappa} \nu^{-2j} \sum_{n=1}^{\nu} n^{2j-1} E_{n-1}(f),$$

where C_1, C_2 depend on κ only.

It remains to transform back to the algebraic case. Equation (7), Lemma 4, and the discussion immediately preceding it show that the algebraic polynomial r_ν given by $r_\nu(x) = (\tilde{t}_\nu + \tilde{p}_\nu)(\arccos x)$ interpolates to f and its first κ derivatives at $t_1, \ldots, t_\gamma$. Also $\| f - r_\nu \|_{[-1,1]} = \| g - (\tilde{t}_\nu - \tilde{p}_\nu) \|_{[-\pi,\pi]}$. Therefore the sequence of polynomials $\{r_\nu\}_{\nu=\nu_1}^{\infty}$ provides the estimate of the present theorem.

Case 2. *None of the interpolation nodes is* ± 1. In this case the Theorem is an obvious corollary of Theorem 2 and Theorem 6.

The most interesting special case is

COROLLARY 9. *If* $f \in C^\kappa[-1, 1]$ *then*

$$E_\nu(f) = O(\nu^{-\beta}) \Rightarrow E_\nu(f, A_\kappa) = O(\nu^{-\beta}) \tag{8}$$

unless β *is an even integer,* $\kappa < \beta \leqslant 2\kappa$, f *is not in* $C^{2\kappa}[-1, 1]$ *and one of the nodes of interpolation is* ± 1 *in which case*

$$E_\nu(f) = O(\nu^{-\beta}) \Rightarrow E_\nu(f, A_\kappa) = O(\nu^{-\beta} \log \nu). \tag{9}$$

Proof. If $\beta \leqslant \kappa$ the result is a corollary of Theorem 1. If $f \in C^{2\kappa}[-1, 1]$, or ± 1 are not nodes of interpolation, the result follows from a combination of Corollary 5 and Corollary 3, respectively, with the estimates of Theorem 6(i). If $\beta > \kappa$ is not an integer then the result is immediate from Theorem 8.

It remains to consider the case of $\beta > \kappa$, an integer. If $\beta \geqslant 2\kappa + 1$ then $k = 2\kappa$ and the result is immediate from Theorem 8. If $\kappa < \beta \leqslant 2\kappa$ then we may assume, without loss of generality, that β is the largest integer in $(\kappa, 2\kappa]$ for which $E_\nu(f) = O(\nu^{-\beta})$. Then by Theorem 6(i) and well-known Jackson theorems it follows that $\beta \geqslant k$. Indeed since $\sum n^{-2}$ converges either $\beta = k$ or $\beta = k + 1$. The first term on the right-hand side of the inequality of Theorem 8 will be $o(\nu^{-\beta})$ if $\beta = k$, and $O(\nu^{-\beta})$ if $\beta = k + 1$. The second term on the right-hand side of this inequality will be $O(\nu^{-\beta})$, unless $\beta = k + 1 = 2j \leqslant 2\kappa$, in which case it will be $O(\nu^{-\beta} \log \nu)$. This concludes the proof.

Remarks. It is not known if the estimate (9) is sharp. It is clear that if the interpolation at the endpoints is restricted to orders $\leqslant [\kappa/2]$ then the last term in the estimate of Theorem 8 would not occur ($\beta \quad \kappa$) and hence there would be no exceptional case.

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